

# Understanding Pseudo-similarity in Graphs: a Path to Proving the Reconstruction Conjecture

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# 1 Introduction

Two vertices  $u$  and  $v$  in a graph  $\Gamma$  are called *pseudo-similar* if the vertex-induced subgraphs  $\Gamma - u$  and  $\Gamma - v$  are isomorphic, but no automorphism of  $\Gamma$  maps  $u$  to  $v$ . Pseudo-similarity was discovered during attempts to prove the long-standing *Graph Reconstruction Conjecture*, of Kelly and Ulam [Kel57, Ula60]. Given a graph  $\Gamma$ , we define the *deck* of  $\Gamma$ , denoted  $\text{Deck}(\Gamma)$ , to be the multiset of isomorphism classes of the graphs  $\Gamma - u$  for each  $u \in V\Gamma$ . The Reconstruction Conjecture claims that every finite graph of order at least 3 is uniquely reconstructible from its deck; that is, given any two finite graphs  $\Gamma$  and  $\Gamma'$  on at least 3 vertices,  $\text{Deck}(\Gamma) \cong \text{Deck}(\Gamma')$  if and only if  $\Gamma \cong \Gamma'$ . The Reconstruction Conjecture has resisted traditional algebraic methods for a considerable amount of time [NW78]. In particular, it is claimed in [HP66, Sto81] that a ‘proof’ of the Reconstruction Conjecture emerged in the 1960s which relied heavily on the non-existence of pseudo-similar vertices. This suggests that a valid proof of the Reconstruction Conjecture may be found through better understanding finite graphs which contain pseudo-similar vertices.

Alongside its study in connection with the Reconstruction Conjecture, pseudo-similarity has attracted interest for study in its own right, for example in [GK82, GK83, Lau97]. In particular, finding the size of the largest possible set of pseudo-similar vertices that a graph can contain has been the subject of intense research efforts [KP75, KJSS81], and the problem of determining the orders of the smallest graphs  $\Gamma_k$  each containing a set of mutually pseudo-similar vertices is still an open problem.

Both of these problems are closely related to the study of isomorphisms between induced subgraphs of a graph  $\Gamma$ . In [JJSS21], it is argued that the study of the *inverse monoid of partial automorphisms* of a graph  $\Gamma$  may be key to breakthroughs in problems like the Reconstruction Conjecture or understanding pseudo-similar vertices. A *partial automorphism* of a graph  $\Gamma$  is an isomorphism between its vertex-induced subgraphs, and the partial automorphisms of a graph  $\Gamma$  form an inverse monoid under composition of partial maps, denoted  $\text{PAut}(\Gamma)$ . This monoid may be viewed as a generalisation of the automorphism group of  $\Gamma$ , as  $\text{PAut}(\Gamma)$  contains  $\text{Aut}(\Gamma)$  as a subgroup.  $\text{PAut}(\Gamma)$  is a rich source of additional information about the algebraic structure of  $\Gamma$ . For example, while many non-isomorphic graphs may have the same automorphism group (even more so, in [ER63] it is proven that almost all graphs have a trivial automorphism group), for any two finite graphs  $\Gamma$  and  $\Gamma'$ , we have  $\text{PAut}(\Gamma) \cong \text{PAut}(\Gamma')$  if and only if  $\Gamma \cong \Gamma'$  or  $\bar{\Gamma} \cong \bar{\Gamma}'$ , where  $\bar{\Gamma}$  denotes the complement of  $\Gamma$ .

In this project, we study how the algebraic theory of inverse monoids of partial automorphisms of graphs began in [JJSS21] can be applied to understanding pseudo-similar vertices in graphs, with a particular emphasis on graphs with sets of mutually pseudo-similar vertices. These results are used to improve the best known upper bound on the size of a set of mutually pseudo-similar vertices in a graph  $\Gamma$ . Finally, we consider digraphs which contain pseudo-similar vertices. In general, it is easier to construct digraphs with pseudo-similar vertices than graphs. The falsity of the Reconstruction Conjecture for digraphs seems to support the connection between pseudo-similarity and graph reconstruction; indeed, the existence of certain families of digraphs with large sets of pseudo-similar vertices is key in the construction of infinite families of counterexamples to the digraph Reconstruction Conjecture [Sto77, Sto81]. We will find that the subtleties which exist for digraphs cause many of the results we prove for graphs to fail for digraphs.

## 1.1 Prerequisites from graph theory

A *graph*  $\Gamma$  is an ordered pair  $(V\Gamma, E\Gamma)$  where  $V\Gamma$  is the *vertex set* and  $E\Gamma \subseteq [V\Gamma]^2$  is the *edge set*, which is a collection of 2-element subsets of  $V\Gamma$ . A *digraph*  $\Gamma$  is an ordered pair  $(V\Gamma, E\Gamma)$  where  $V\Gamma$  is the *vertex set* and  $E\Gamma \subseteq V\Gamma \times V\Gamma$  is the *edge set*, which is a collection of ordered pairs of elements of  $V\Gamma$ . Digraphs may have loops, but have no multiple edges. All graphs and digraphs considered are finite. For more detail on general graph theory, see [Die24].

If  $\Gamma$  is a graph, and the vertices  $u$  and  $v$  are adjacent, then we write  $u \sim v$ . The *neighbourhood* of the vertex  $u$  in  $\Gamma$  is denoted by  $N_\Gamma(u)$ . The *degree* of the vertex  $u$  in  $\Gamma$  is written as  $\deg_\Gamma(u)$ . The *degree sequence* of  $\Gamma$  is the list of all the degrees of the vertices in  $\Gamma$ , with multiplicity, written in increasing order.

If  $\Gamma$  is a digraph, we write  $u \rightarrow v$  to mean there is a directed edge from the vertex  $u$  to the vertex  $v$ . The *in-neighbourhood* of  $u$  in  $\Gamma$  is written as  $N_{\Gamma \rightarrow}(u)$ . Similarly, the *out-neighbourhood* of  $u$  in  $\Gamma$  is written as  $N_{\Gamma \leftarrow}(u)$ . If  $\Gamma$  is a digraph, we write  $N_\Gamma(u)$  to mean  $N_{\Gamma \rightarrow}(u) \cup N_{\Gamma \leftarrow}(u)$ . The *extended neighbourhood* of  $u$  in  $\Gamma$  is the set  $N'_\Gamma(u) = N_\Gamma(u) \cup \{u\}$ .

An *isomorphism*  $\varphi$  between two graphs  $\Gamma, \Gamma'$  is a bijection

$$\varphi : V\Gamma \rightarrow V\Gamma'$$

which preserves both edges and non-edges. We denote isomorphism by  $\Gamma \cong \Gamma'$ . An *automorphism* is an isomorphism from a graph to itself. The group of automorphisms of  $\Gamma$  is denoted by  $\text{Aut}(\Gamma)$ .

If  $\Gamma$  is a graph or a digraph, then the *induced subgraph* of  $\Gamma$  with vertex set  $A \subseteq V\Gamma$  is denoted by  $\Gamma[A]$ . Given an isomorphism

$$\varphi : \Gamma[A] \rightarrow \Gamma[B]$$

between induced subgraphs of  $\Gamma$ , we abuse notation and write

$$\varphi : A \rightarrow B$$

where this will not cause confusion.

If  $u \in V\Gamma$ , the induced subgraph  $\Gamma[V\Gamma \setminus \{u\}]$  is denoted by  $\Gamma - u$ . If  $u, v \in V\Gamma$ , the induced subgraph  $\Gamma[V\Gamma \setminus \{u, v\}]$  is denoted by  $\Gamma - u - v$ .

We define the following two equivalence relations on  $V\Gamma$ .

1. *Similarity*: If  $u, v \in V\Gamma$ , then  $u$  and  $v$  are called *similar* if there is a  $\varphi \in \text{Aut}(\Gamma)$  such that  $u\varphi = v$ . This is denoted by  $u \sim_S v$ .
2. *Removal-similarity*: If  $u, v \in V\Gamma$ , then  $u$  and  $v$  are called *removal-similar* if  $\Gamma - u \cong \Gamma - v$ . This is denoted by  $u \sim_R v$ .

**Definition 1.1** (Pseudo-similarity). Let  $\Gamma$  be a graph (or a digraph), and let  $u, v \in V\Gamma$ . Then  $u$  and  $v$  are called *pseudo-similar* if they are removal-similar but not similar. That is, if  $u \sim_R v$  but not  $u \sim_S v$ .

While similarity and removal-similarity are both equivalence relations, pseudo-similarity is *not* an equivalence relation. It is irreflexive (since a vertex is always similar to itself, by the identity

automorphism), and is not, in general, transitive. Therefore, in order to study sets of vertices which are mutually pseudo-similar, the following definition is made.

**Definition 1.2** (Mutually pseudo-similar vertices). Let  $\Gamma$  be a graph (or a digraph), and let  $A = \{u_1, \dots, u_k\} \subseteq V\Gamma$ , with  $k \geq 2$ . The vertices in  $A$  are called *mutually pseudo-similar* if they are all pairwise pseudo-similar.

In [Lau97], it is observed that many of the most interesting questions about pseudo-similarity in graphs are concerned with sets of mutually pseudo-similar vertices, and these questions have attracted the attention of many authors. Mutually pseudo-similar vertices also appear to be the most enigmatic: while existence of pairs of pseudo-similar vertices has an explanation in terms of a kind of truncation of cyclic symmetry [GK82] no such explanation exists for larger sets of mutually pseudo-similar vertices. Much of the work in this essay is concerned with finding necessary and sufficient conditions for sets of mutually pseudo-similar vertices to exist. In particular, we consider the following question, first asked in [KJSS81] and most recently in [JJSS21].

**Problem 1.3.** *For each  $k$ , what is the order of the smallest graph  $\Gamma_k$  containing a set of  $k$  mutually pseudo-similar vertices?*

## 1.2 From the automorphism group to the inverse monoid of partial automorphisms

A *partial automorphism* of a graph  $\Gamma$  is an isomorphism between two of its vertex-induced subgraphs. The partial automorphisms of  $\Gamma$  form an *inverse monoid* under the operation of composition of partial maps. This monoid is called the *inverse monoid of partial automorphisms* of  $\Gamma$ , or the *partial automorphism monoid* of  $\Gamma$ , and is denoted  $\text{PAut}(\Gamma)$ . We denote the domain of a partial automorphism  $\varphi$  by  $\text{dom}(\varphi)$ , and the image by  $\text{im}(\varphi)$ . The *rank* of a partial automorphism is the size of its domain.

We are interested in two aspects of the algebraic structure of such monoids; namely, Green's relations and the natural partial order, which we shall review. For a good source on general semigroup theory, see [How95]. The theory we review here is developed in more detail in [JJSS21]. We use the cycle-and-path notation from [JJSS21] to represent partial permutations, however we write mappings on the *right* of their arguments, and compose maps from *left to right*. This, in particular, has the effect of swapping the roles of the  $\mathcal{L}$  and  $\mathcal{R}$ -relations. We denote composition of partial maps by  $\circ$ .

For a monoid  $M$ , Green's relations are a collection of equivalence relations defined on  $M$  which describe divisibility in  $M$ . These are  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $\mathcal{H}$ ,  $\mathcal{D}$ , and  $\mathcal{J}$ ; the final two of which coincide in the case  $M$  is finite. For  $a, b \in M$ , the  $\mathcal{L}$ -relation is defined by

$$a\mathcal{L}b \text{ if there exist } x, y \in M \text{ such that } xa = b \text{ and } yb = a.$$

Dually, the  $\mathcal{R}$ -relation is defined by

$$a\mathcal{R}b \text{ if there exist } x, y \in M \text{ such that } ax = b \text{ and } by = a.$$

The  $\mathcal{H}$ -relation is defined by  $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ . That is,  $a\mathcal{H}b$  if and only if  $a\mathcal{L}b$  and  $a\mathcal{R}b$ . The  $\mathcal{D}$ -relation is the largest equivalence relation containing both the  $\mathcal{L}$  and  $\mathcal{R}$ -relations. For  $a, b \in M$ ,

$$a\mathcal{D}b \text{ if there exists } x \in M \text{ such that } a\mathcal{L}x\mathcal{R}b.$$

The final relation,  $\mathcal{J}$  is equal to  $\mathcal{D}$  in the case when  $M$  is finite (Chapter 2, [How95]). We are interested in studying Green's relations on the partial automorphism monoid of a graph. When  $\Gamma$  is a finite graph,  $\text{PAut}(\Gamma)$  is also finite, therefore we will not need to consider the  $\mathcal{J}$ -relation.

To study pseudo-similarity in graphs, we need to understand which graph-theoretic properties are described by Green's relations on  $\text{PAut}(\Gamma)$ . The following proposition is from [JJSS21].

**Proposition 1.4.** *Let  $\Gamma$  be a graph, and  $\varphi, \psi \in \text{PAut}(\Gamma)$ . Then*

1.  $\varphi \mathcal{L} \psi$  if and only if  $\text{im}(\varphi) = \text{im}(\psi)$ .
2.  $\varphi \mathcal{R} \psi$  if and only if  $\text{dom}(\varphi) = \text{dom}(\psi)$ .
3.  $\varphi \mathcal{H} \psi$  if and only if  $\text{dom}(\varphi) = \text{dom}(\psi)$  and  $\text{im}(\varphi) = \text{im}(\psi)$ .
4.  $\varphi \mathcal{D} \psi$  if and only if  $\text{dom}(\varphi) \cong \text{dom}(\psi)$ .

*Proof.* Let  $\Gamma$  be a graph, and let  $\varphi, \psi \in \text{PAut}(\Gamma)$ .

1. Suppose  $\varphi \mathcal{L} \psi$ . Then, there exist  $\alpha, \beta \in \text{PAut}(\Gamma)$  such that  $\alpha \circ \varphi = \psi$  and  $\beta \circ \psi = \varphi$ . Since  $\text{im}(\alpha \circ \varphi) \subseteq \text{im}(\varphi)$ , it follows that  $\text{im}(\psi) \subseteq \text{im}(\varphi)$ . The other containment comes from the symmetric argument considering  $\beta \circ \psi$ . Now suppose  $\text{im}(\varphi) = \text{im}(\psi)$ . Then,  $(\psi \circ \varphi^{-1}) \circ \varphi = \psi \circ (\varphi^{-1} \circ \varphi) = \psi \circ \text{id}_{\text{im}(\varphi)} = \psi$ . The other product follows from swapping the roles of  $\varphi$  and  $\psi$ . Hence  $\varphi \mathcal{L} \psi$ .
2. This argument is dual to the argument in (1).
3. Since  $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ , it follows that  $\varphi \mathcal{H} \psi$  if and only if  $\varphi \mathcal{L} \psi$  and  $\varphi \mathcal{R} \psi$ . Therefore,  $\varphi \mathcal{H} \psi$  if and only if  $\text{im}(\varphi) = \text{im}(\psi)$  and  $\text{dom}(\varphi) = \text{dom}(\psi)$ .
4.  $\varphi \mathcal{D} \psi$  if and only if there exists  $\alpha \in \text{PAut}(\Gamma)$  such that  $\varphi \mathcal{L} \alpha \mathcal{R} \psi$ . The first relation holds if and only if  $\text{im}(\varphi) = \text{im}(\alpha)$ , and the latter relation if and only if  $\text{dom}(\alpha) = \text{dom}(\psi)$ . Hence,  $\varphi \mathcal{D} \psi$  if and only if there exists an isomorphism  $\alpha : \text{dom}(\psi) \rightarrow \text{im}(\varphi)$ . This is equivalent to  $\text{dom}(\psi) \cong \text{im}(\varphi)$ , or equivalently,  $\text{dom}(\psi) \cong \text{dom}(\varphi)$ .

□

Note that the roles of the  $\mathcal{L}$  and  $\mathcal{R}$  are swapped in this report compared to [JJSS21]. This is simply because we write partial maps on the right rather than on the left.

*Remark.* Proposition 1.4 is also true if we replace  $\Gamma$  with a digraph. The proof is the same.

Now we define the second aspect of the structure of  $\text{PAut}(\Gamma)$  which will be required to study pseudo-similarity: the *natural partial order* on  $\text{PAut}(\Gamma)$ . This order is defined using the collection of partial identity maps on  $\Gamma$ . Given any induced subgraph  $\Gamma[A]$  of  $\Gamma$ , the identity map on  $\Gamma[A]$  is a partial automorphism, and is denoted  $\text{id}_A$ . For  $\varphi, \psi \in \text{PAut}(\Gamma)$ , we define  $\varphi \leq \psi$  if and only if there exists an identity map  $\text{id}_A$  such that  $\varphi = \text{id}_A \circ \psi$ . It is verified in [JJSS21] that this indeed defines a partial order on  $\text{PAut}(\Gamma)$ .

This ordering is equal to the restriction ordering on  $\text{PAut}(\Gamma)$ : for any  $\varphi, \psi \in \text{PAut}(\Gamma)$ , we have  $\varphi \leq \psi$  if and only if  $\varphi$  is obtained by restricting the domain of  $\psi$  [JJSS21]. Where this is the case,

$\psi$  will be called an *extension* of  $\varphi$ , and  $\varphi$  is a partial automorphism *induced* by  $\psi$ . An element  $\varphi \in \text{PAut}(\Gamma)$  is called *maximal* in the partial order if, for any  $\psi \in \text{PAut}(\Gamma)$ , we have

$$\varphi \leq \psi \implies \varphi = \psi.$$

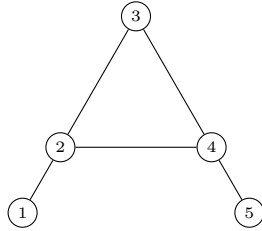
That is, a partial automorphism is maximal if it cannot be extended to a partial automorphism with a larger domain. Maximal partial automorphisms are central to describing pseudo-similarity.

## 2 Results of an Algebraic Flavour

### 2.1 An alternate algebraic description of pseudo-similarity

In this section, we use tools from inverse semigroup theory to provide a different characterisation of pairs of pseudo-similar vertices, and sets of mutually pseudo-similar vertices to the standard one. As one may expect after reading the previous section, these tools are precisely Green's relations and the natural partial order on  $\text{PAut}(\Gamma)$ .

It may first seem the case that vertices  $u, v$  are pseudo-similar in the graph  $\Gamma$  if there exists a partial automorphism  $\varphi : \Gamma - u \rightarrow \Gamma - v$  which cannot be extended to an automorphism of  $\Gamma$  mapping  $u$  to  $v$ . This is, however, not sufficient, as shown in [Example 2.1](#).



$\Gamma$

Figure 1: The graph  $\Gamma$ .

**Example 2.1.** Consider the graph  $\Gamma$  in [Figure 1](#). The map

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & - \end{pmatrix}$$

is a partial automorphism of  $\Gamma$  of rank  $|V\Gamma| - 1$  which maps the graph  $\Gamma - 5$  to the graph  $\Gamma - 1$ .  $\varphi$  cannot be extended to an automorphism of  $\Gamma$ , as the only permutation which extends  $\varphi$  is

$$\varphi' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

which maps the edge  $\{1, 2\}$  onto the non-edge  $\{5, 3\}$ . However, the vertices 1 and 2 are not pseudo-similar in  $\Gamma$ . The map

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

is an automorphism of  $\Gamma$  mapping 5 to 1, therefore  $1 \sim_S 5$  in  $\Gamma$ .

The correct condition is stronger, and involves Green's  $\mathcal{H}$ -relation in  $\text{PAut}(\Gamma)$ .

**Theorem 2.2** (Pseudo-similarity). *Let  $\Gamma$  be a graph with removal-similar vertices  $u, v \in V\Gamma$ . Then the following are equivalent:*

1. *The vertices  $u$  and  $v$  are pseudo-similar in  $\Gamma$ .*
2. *The  $\mathcal{H}$ -class  $H_\varphi$  of any partial automorphism  $\varphi : \Gamma - u \rightarrow \Gamma - v$  has the property that every  $\psi \in H_\varphi$  is maximal in the natural partial order on  $\text{PAut}(\Gamma)$ .*

*Proof.* Let  $\Gamma$ ,  $u$  and  $v$  be as given in the statement of the theorem. Suppose first that  $u$  and  $v$  are pseudo-similar in  $\Gamma$ . By assumption,  $u \sim_R v$ , so there exists a partial automorphism of  $\Gamma$

$$\varphi : \Gamma - u \rightarrow \Gamma - v.$$

Suppose, for a contradiction, that there exists  $\psi \in H_\varphi$  which is not maximal in the natural partial order on  $\text{PAut}(\Gamma)$ . By [Proposition 1.4](#),  $\text{dom}(\psi) = \Gamma - u$ , and  $\text{im}(\psi) = \Gamma - v$ , and so

$$\psi : \Gamma - u \rightarrow \Gamma - v.$$

Therefore, any extension  $\psi'$  of  $\psi$  must be of the form  $\psi \vee (uv)$ , and so maps  $u$  to  $v$ . Since  $\psi$  is not maximal in the natural partial order on  $\text{PAut}(\Gamma)$ , it follows that  $\psi' \in \text{PAut}(\Gamma)$ . Therefore, there is an automorphism  $\psi'$  of  $\Gamma$  which maps  $u$  to  $v$ . It follows that  $u \sim_S v$  in  $\Gamma$ . This is a contradiction. Therefore, we must conclude that every  $\psi \in H_\varphi$  is maximal in the partial order on  $\text{PAut}(\Gamma)$ .

For the backward direction, let

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

be a partial automorphism. Such a  $\varphi$  exists by the assumption that  $u \sim_R v$ . Suppose that every  $\psi \in H_\varphi$  is maximal in the partial order on  $\text{PAut}(\Gamma)$ . We claim that no automorphism of  $\Gamma$  maps  $u$  to  $v$ . Suppose for a contradiction that  $\theta \in \text{Aut}(\Gamma)$  is such that  $u\theta = v$ . Then,  $\theta$  induces a partial automorphism

$$\theta' : \Gamma - u \rightarrow \Gamma - v.$$

By [Proposition 1.4](#),  $\theta' \in H_\varphi$ . However,  $\theta' = \text{id}_{\Gamma-u} \circ \theta$ , and so  $\theta' \leq \theta$  in the partial order on  $\text{PAut}(\Gamma)$ . This contradicts the assumed maximality of every  $\psi \in H_\varphi$ . Therefore, if every  $\psi \in H_\varphi$  is maximal in the natural partial order on  $\text{PAut}(\Gamma)$ , then  $u$  and  $v$  are not similar. It follows that they are pseudo-similar.  $\square$

Following [Theorem 2.2](#), it seems appropriate to seek a characterisation of sets of mutually pseudo-similar vertices in terms of a graph's inverse monoid of partial automorphisms. Here we encounter the problem that pseudo-similarity is not an equivalence relation: it is irreflexive, and importantly, is not transitive. This is illustrated in examples [2.3](#) and [2.4](#). In contrast, Green's relations are all equivalence relations, which presents a difficulty in obtaining the desired characterisation. Given a

$\mathcal{D}$ -class  $D$  of rank  $|V\Gamma| - 1$  elements in  $\text{PAut}(\Gamma)$ , there may exist partial automorphisms in  $D$  (with their domain not equal to their image) which can be extended to automorphisms of  $\Gamma$ , and still, there may exist other partial automorphisms in  $D$  which cannot be extended to automorphisms of  $\Gamma$ . Two examples of this are given, one for digraphs and one for graphs; [Figure 2](#) and [Figure 3](#) respectively.

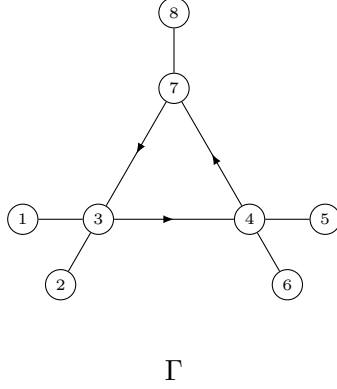


Figure 2: The digraph  $\Gamma$ .

**Example 2.3.** Consider the digraph  $\Gamma$  in [Figure 2](#). The vertex 1 is pseudo-similar to the vertex 5, and the vertex 5 is pseudo-similar to the vertex 2. However, the vertices 1 and 2 are similar, and therefore are not pseudo-similar, showing that pseudo-similarity is not transitive in digraphs.

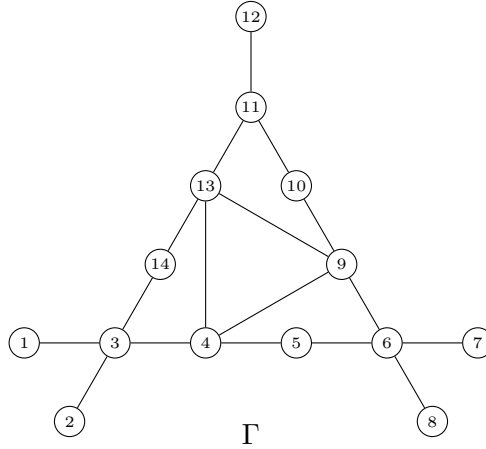


Figure 3: The graph  $\Gamma$ .

**Example 2.4.** Consider the graph  $\Gamma$  in [Figure 3](#). The vertex 1 is pseudo-similar to the vertex 7, and the vertex 7 is pseudo-similar to the vertex 2. However, the vertices 1 and 2 are similar, and therefore are not pseudo-similar, showing that pseudo-similarity is also not transitive in graphs.

For a set  $A \subseteq V\Gamma$  to be mutually pseudo-similar, we must require that the conditions in [Theorem 2.2](#) hold *pairwise* between the vertices in  $A$ . First, for concision, the following definition is made.

**Definition 2.5.** Given a graph  $\Gamma$ , and an  $\mathcal{H}$ -class  $H$  in  $\text{PAut}(\Gamma)$ , then we say that  $H$  has the *element maximality property* if every  $\varphi \in H$  is maximal in the natural partial order on  $\text{PAut}(\Gamma)$ .

**Theorem 2.6** (Mutually pseudo-similar vertices). *Let  $\Gamma$  be a graph, and let  $A = \{u_1, u_2, \dots, u_k\} \subseteq V\Gamma$ . Then the vertices in  $A$  are mutually pseudo-similar if and only if*

1. *The identity maps  $\text{id}_{\Gamma-u_1}, \text{id}_{\Gamma-u_2}, \dots, \text{id}_{\Gamma-u_k}$  are all  $\mathcal{D}$ -related.*
2. *For any pair  $u_i, u_j \in A$  of distinct vertices, the  $\mathcal{H}$ -class of any partial automorphism  $\varphi : \Gamma - u_i \rightarrow \Gamma - u_j$  has the element maximality property.*

*Proof.* First suppose that  $\Gamma$  contains a set  $A = \{u_1, u_2, \dots, u_k\} \subseteq V\Gamma$  of vertices which are mutually pseudo-similar. Let  $u_i, u_j \in A$ , where  $i \neq j$ . By assumption,  $u_i$  and  $u_j$  are pseudo-similar, so are both removal-similar and not similar. Since  $u_i$  and  $u_j$  are removal-similar,  $\Gamma - u_i \cong \Gamma - u_j$  and therefore  $\text{id}_{\Gamma-u_i} \mathcal{D} \text{id}_{\Gamma-u_j}$  by [Proposition 1.4](#). Moreover, since  $u_i$  and  $u_j$  are pseudo-similar, [Theorem 2.2](#) implies that the  $\mathcal{H}$ -class of any partial automorphism

$$\varphi : \Gamma - u_i \rightarrow \Gamma - u_j$$

has the element maximality property. Since  $u_i$  and  $u_j$  were chosen arbitrarily, it follows that conditions (1) and (2) hold. This completes the forward direction.

Now suppose that  $\Gamma$  contains a set  $A = \{u_1, u_2, \dots, u_k\} \subseteq V\Gamma$  which satisfies conditions (1) and (2). Let  $u_i, u_j \in A$ , where  $u_i \neq u_j$ . By the assumption in condition (1),  $\text{id}_{\Gamma-u_i} \mathcal{D} \text{id}_{\Gamma-u_j}$ . Moreover, by [Proposition 1.4](#),

$$\text{dom}(\text{id}_{\Gamma-u_i}) \cong \text{dom}(\text{id}_{\Gamma-u_j}),$$

so  $\Gamma - u_i \cong \Gamma - u_j$ . Therefore  $u_i$  and  $u_j$  are removal-similar.

By (2), the  $\mathcal{H}$ -class of any partial automorphism

$$\varphi : \Gamma - u_i \rightarrow \Gamma - u_j$$

has the element maximality property, and so, by [Theorem 2.2](#),  $u_i$  and  $u_j$  are pseudo-similar. Since  $u_i$  and  $u_j$  were chosen arbitrarily, the vertices in  $A$  are pairwise pseudo-similar, hence are mutually pseudo-similar. □

*Remark.* Theorems [2.2](#) and [2.6](#) are also true for digraphs. The proofs are identical.

[Theorem 2.6](#) is limited in the sense that it is essentially [Theorem 2.2](#) applied pairwise to the vertices in the set  $A$ . If these methods are to be used to provide a resolution to [Question 1.3](#), then a stronger characterisation of mutually pseudo-similar vertices should be found. What is needed is a more complete theory of how Green's relations interact with the natural partial order on an inverse monoid  $M$ , in particular in the case when  $M$  is the partial automorphism monoid of a graph or a digraph.

## 2.2 Extending partial automorphisms between vertex-deleted subgraphs to automorphisms

This section considers when a partial automorphism of rank  $|V\Gamma| - 1$  in a graph  $\Gamma$  can be extended to an automorphism of  $\Gamma$ .

**Lemma 2.7.** *Let  $\Gamma$  be a graph and  $u, v \in V\Gamma$  be vertices. Let  $\varphi : \Gamma - u \rightarrow \Gamma - v$  be a partial automorphism of  $\Gamma$ . Then  $\varphi$  can be extended to an automorphism of  $\Gamma$  if and only if  $\varphi$  induces a partial automorphism*

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

*Proof.* Let  $\Gamma$ ,  $u$ ,  $v$ , and  $\varphi$  be as given in the statement of the lemma. Suppose  $\varphi$  induces a partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Since  $\varphi : \Gamma - u \rightarrow \Gamma - v$ , the only extension of  $\varphi$  is the function  $\varphi' = \varphi \vee (uv]$  which maps  $u$  to  $v$ . A permutation of  $V\Gamma$  is an automorphism of  $\Gamma$  if and only if it preserves both the edge and non-edge relations on  $V\Gamma$ ; since  $\varphi'$  is an extension of the partial automorphism  $\varphi$ , it follows that  $\varphi'$  is an automorphism of  $\Gamma$  if and only if

$$u \sim w \text{ if and only if } v \sim w\varphi, \text{ for all } w \in V\Gamma.$$

Let  $w \in V\Gamma$ . Then  $u \sim w$  if and only if  $w \in N_\Gamma(u)$ . Since  $\varphi$  induces the partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

it follows that  $w\varphi \in N_\Gamma(v)$  if and only if  $w \in N_\Gamma(u)$ . Hence,  $u \sim w$  if and only if  $v \sim w\varphi$ . Therefore,  $\varphi'$  is an automorphism of  $\Gamma$ . This completes the forward direction.

For the backward direction, suppose that  $\varphi$  does not induce a partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Then, there exists  $w \in N_\Gamma(u)$  such that  $w\varphi \notin N_\Gamma(v)$ . Therefore,  $w$  is such that  $u \sim w$  but not  $v \sim w\varphi$ , and so  $\varphi'$  is not an automorphism of  $\Gamma$ . This completes the backward direction.  $\square$

**Theorem 2.8.** *Let  $\Gamma$  be a graph with vertices  $u, v \in V\Gamma$ . Suppose that  $\varphi : \Gamma - u \rightarrow \Gamma - v$  is a partial automorphism of  $\Gamma$ , and let  $A$  denote the image of  $N_\Gamma(u)$  under  $\varphi$ . Then,  $u$  and  $v$  are similar in  $\Gamma$  if and only if  $\Gamma - v$  has an automorphism mapping  $A$  to  $N_\Gamma(v)$ .*

*Proof.* Let  $\Gamma$ ,  $u$ ,  $v$ ,  $\varphi$  and  $A$  be as given in the statement of the theorem. Suppose first that  $\theta$  is an automorphism of  $\Gamma - v$  such that  $A\theta = N_\Gamma(v)$ . We will show that  $u \sim_S v$  in  $\Gamma$ . Consider the partial automorphism  $\psi = \varphi \circ \theta$ . Since

$$\varphi : \Gamma - u \rightarrow \Gamma - v,$$

and

$$\theta : \Gamma - v \rightarrow \Gamma - v,$$

it follows that  $\text{dom}(\psi) = \Gamma - u$ , and  $\text{im}(\psi) = \Gamma - v$ . Therefore,

$$\psi : \Gamma - u \rightarrow \Gamma - v.$$

Now, consider the image of  $N_\Gamma(u)$  under  $\psi$ . From the definition of  $\psi$ ,

$$\begin{aligned} (N_\Gamma(u))\psi &= (N_\Gamma(u))(\varphi\theta) \\ &= ((N_\Gamma(u))\varphi)\theta \\ &= A\theta \\ &= N_\Gamma(v). \end{aligned}$$

It follows that  $\psi$  induces a partial automorphism

$$\alpha : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and hence, by [Lemma 2.7](#),  $\psi$  can be extended to an automorphism of  $\Gamma$  mapping  $u$  to  $v$ . Therefore,  $u \sim_S v$  in  $\Gamma$ .

Now suppose that  $u \sim_S v$  in  $\Gamma$ . We will show that  $\Gamma - v$  has an automorphism mapping  $A$  to  $N_\Gamma(v)$ . Since  $u$  and  $v$  are similar in  $\Gamma$ , there exists an automorphism  $\alpha$  of  $\Gamma$  such that  $u\alpha = v$ . Therefore, restricting  $\alpha$  induces a partial automorphism

$$\psi : \Gamma - u \rightarrow \Gamma - v.$$

Moreover, it follows, since  $\psi$  is induced from by automorphism of  $\Gamma$ , that  $\psi$  induces a partial automorphism

$$\beta : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Consider the partial automorphism  $\theta = \varphi^{-1}\psi$ . Notice that, since

$$\varphi^{-1} : \Gamma - v \rightarrow \Gamma - u,$$

and

$$\psi : \Gamma - u \rightarrow \Gamma - v,$$

it follows that  $\text{dom}(\theta) = \text{im}(\theta) = \Gamma - v$ . Now consider the image of  $A$  under  $\theta$ . From the definition of  $\theta$ ,

$$\begin{aligned} A\theta &= A(\varphi^{-1}\psi) \\ &= (A\varphi^{-1})\psi \\ &= (N_\Gamma(u))\psi \\ &= N_\Gamma(v). \end{aligned}$$

Therefore,  $\theta$  is an automorphism of  $\Gamma - v$  mapping  $A$  onto  $N_\Gamma(v)$ . This proves our claim. □

### 3 Results of a Combinatorial Flavour

#### 3.1 Upper bounds on the size of a set of mutually pseudo-similar vertices

Here, we apply the algebraic results of the previous section to obtain upper bounds on the size of a set of mutually pseudo-similar vertices in a graph of a given order. The proofs of [Lemma 3.1](#) and [Theorem 3.2](#) can be found in [\[Lau97\]](#); we recount those proofs here, rephrased in the language of partial automorphisms, in order to illustrate the kinds of arguments used in the longer proof of [Theorem 3.4](#). The statement of [Theorem 3.3](#) is also mentioned in [\[Lau97\]](#), however a proof is not given. We provide a proof. We then extend these results to [Theorem 3.4](#).

**Notation.** If  $f, g : \mathbb{N} \rightarrow \mathbb{R}$  are real sequences, then the notation  $f(n) = \mathcal{O}(g(n))$  means there exists a real number  $C > 0$  and  $N \in \mathbb{N}$  such that  $|f(n)| \leq C |g(n)|$  for all  $n \geq N$ .

The following lemma is an important observation in characterising graphs with sets of mutually pseudo-similar vertices, and follows directly from [Lemma 2.7](#).

**Lemma 3.1.** *Let  $\Gamma$  be a regular graph. Then  $\Gamma$  does not contain a pair of pseudo-similar vertices.*

*Proof.* Suppose that the degree of any vertex in  $\Gamma$  is  $k$ . If  $u \in V\Gamma$  then the set of neighbours  $N_\Gamma(u)$  of  $u$  in  $\Gamma$  all have degree  $k - 1$  in the graph  $\Gamma - u$ . A similar observation is true for any other vertex  $v \in V\Gamma$ . By a degree argument, any partial automorphism  $\varphi : \Gamma - u \rightarrow \Gamma - v$  must induce a partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Therefore, by [Lemma 2.7](#), the partial automorphism  $\varphi$  can be extended to an automorphism  $\varphi'$  of  $\Gamma$  such that  $u\varphi' = v$ . It follows that if  $u, v$  are removal-similar in  $\Gamma$ , then they are similar.  $\square$

**Theorem 3.2.** *A graph  $\Gamma$  cannot have all its vertices mutually pseudo-similar.*

*Proof.* Let  $u, v \in V\Gamma$  be such that  $\Gamma - u \cong \Gamma - v$ . Then  $\deg_\Gamma(u) = \deg_\Gamma(v)$ . It follows that if every vertex in  $\Gamma$  is removal-similar to every other vertex, then  $\Gamma$  is regular. Therefore, by [Lemma 3.1](#),  $\Gamma$  does not contain any pseudo-similar vertices.  $\square$

We can do a little better.

**Theorem 3.3.** *Let  $\Gamma$  be a graph. Then  $\Gamma$  does not contain a set of mutually pseudo-similar vertices of size  $|V\Gamma| - 1$ .*

*Proof.* Let  $\Gamma$  be a graph. Let  $A \subseteq V\Gamma$  be a set of mutually pseudo-similar vertices in  $\Gamma$ , and suppose for a contradiction that  $|A| = |V\Gamma| - 1$ .

Since all the vertices in  $A$  are removal-similar by assumption,  $\deg_\Gamma(u) = k$  for all  $u \in A$  and a fixed  $k$ . If  $w \in V\Gamma \setminus A$  is the remaining vertex, then  $\deg_\Gamma(w) \neq k$  since then  $\Gamma$  would be regular, so cannot contain pseudo-similar vertices by [Lemma 3.1](#). Let  $\deg_\Gamma(w) = r \neq k$ .

We consider 3 cases, depending on the value of  $r$ .

- In the case  $r = 0$ , then  $w$  is an isolated vertex of  $\Gamma$ . It follows that the vertices in  $A$  are mutually pseudo-similar in  $\Gamma$  if and only if they are mutually pseudo-similar in the graph  $\Gamma - w$ . By [Theorem 3.2](#), the vertices in  $A$  cannot be mutually pseudo-similar in  $\Gamma - w$ , and therefore cannot be mutually pseudo-similar in  $\Gamma$ . This is a contradiction.
- In the case  $r = |V\Gamma| - 1$ , consider the complement  $\tilde{\Gamma}$  of  $\Gamma$ . By Corollary 6.5 in [\[JJSS21\]](#),  $\Gamma$  and  $\tilde{\Gamma}$  have the same partial automorphism monoid, hence the vertices in  $A$  are mutually pseudo-similar in  $\Gamma$  if and only if they are mutually pseudo-similar in  $\tilde{\Gamma}$ . If  $r = |V\Gamma| - 1$ , then  $w$  is an isolated vertex in  $\tilde{\Gamma}$ , and so the vertices in  $A$  cannot be pseudo-similar in  $\tilde{\Gamma}$ . This is a contradiction.
- In the case  $0 < r < |V\Gamma| - 1$ , there must exist  $u, v \in A$  such that  $u \notin N_\Gamma(w)$  but  $v \in N_\Gamma(w)$ . We show by a degree argument that  $\Gamma - u \not\cong \Gamma - v$ . First consider  $\Gamma - u$ . Since  $w \notin N_\Gamma(u)$ ,

the degree of  $w$  in  $\Gamma - u$  is  $r$ . Moreover, for any  $w' \in N_\Gamma(u)$ , the vertex  $w'$  has degree  $k - 1$  in  $\Gamma - u$ . It follows that the degree sequence of  $\Gamma - u$  is

$$(r, \underbrace{k-1, \dots, k-1}_k, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}).$$

Now consider  $\Gamma - v$ . Since  $w \in N_\Gamma(v)$ , the degree of  $w$  in  $\Gamma - v$  is  $r - 1$ . Given any  $w' \in N_\Gamma(v) \setminus \{w\}$ , the degree of  $w'$  in  $\Gamma - v$  is  $k - 1$ . Therefore  $\Gamma - v$  has degree sequence

$$(r-1, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-1}).$$

Consider in particular the number of vertices of degree  $k$ . In the case  $r = k + 1$ , the graph  $\Gamma - u$  has  $|V\Gamma| - k - 2$  vertices of degree  $k$ , and  $\Gamma - v$  has  $|V\Gamma| - k$  vertices of degree  $k$ . Therefore, in this case,  $\Gamma - u \not\cong \Gamma - v$ . Moreover, if  $r \neq k + 1$ , then  $\Gamma - u$  has  $|V\Gamma| - k - 2$  vertices of degree  $k$ , and  $\Gamma - v$  has  $|V\Gamma| - k - 1$  vertices of degree  $k$ , so in this case also  $\Gamma - u \not\cong \Gamma - v$ . This is a contradiction. □

It is therefore sensible to ask if this method can be extended to show that a graph  $\Gamma$  cannot contain a set of mutually pseudo-similar vertices of size  $|V\Gamma| - 2$ .

**Theorem 3.4.** *Let  $\Gamma$  be a graph. Then  $\Gamma$  does not contain a set of mutually pseudo-similar vertices of size  $|V\Gamma| - 2$ .*

The proof of [Theorem 3.4](#) is by contradiction. The argument is not simple, and is therefore structured into four main claims about  $\Gamma$  from which the desired contradiction is obtained. These are Propositions (a), (b), (c) and (d).

Suppose that  $\Gamma$  contains a set  $A \subseteq V\Gamma$  of mutually pseudo-similar vertices, and that  $|A| = |V\Gamma| - 2$ . Assume  $|A| \geq 2$ . Let  $w, w' \in V\Gamma \setminus A$  be the two remaining vertices. Then, there exists an integer  $k$  such that  $\deg_\Gamma(u) = k$  for all  $u \in A$ . Notice that  $k \geq 1$ , otherwise the vertices in  $A$  would all be isolated vertices and hence would be similar. Let  $\deg_\Gamma(w) = r$  and  $\deg_\Gamma(w') = s$ .

Our first claim about  $\Gamma$  is the following.

**Proposition (a).** For each  $u \in A$ , either  $u \in N_\Gamma(w)$  or  $N_\Gamma(w')$ , but not both. That is, every  $u \in A$  separates  $w$  and  $w'$ .

*Proof.* We prove this by showing that there cannot exist  $u \in A$  such that both  $u \notin N_\Gamma(w)$  and  $u \notin N_\Gamma(w')$ . The case where both  $u \in N_\Gamma(w)$  and  $u \in N_\Gamma(w')$  follows from the symmetric argument, or by considering the complement  $\tilde{\Gamma}$  of  $\Gamma$ .

Suppose that there exists  $u \in A$  such that  $u \notin N_\Gamma(w)$  and  $N_\Gamma(w')$ . If  $v \notin N_\Gamma(w)$  and  $N_\Gamma(w')$  for all  $v \in A$ , then  $\Gamma$  is not connected, and  $A$  is a connected component of  $\Gamma$ . It follows that the vertices in  $A$  are mutually pseudo-similar in  $\Gamma$  if and only if they are mutually pseudo-similar in the graph  $\Gamma[A]$ . Therefore, by [Theorem 3.2](#), the vertices in  $A$  are not mutually pseudo-similar in  $\Gamma$ .

It follows that there exists  $v \in A$  such that at least one of  $v \in N_\Gamma(w)$  or  $v \in N_\Gamma(w')$  is true. Suppose, without loss of generality, that  $v \in N_\Gamma(w)$ . We show, by a degree argument, that  $v \notin N_\Gamma(w')$ . Suppose that  $v \in N_\Gamma(w')$ . Since  $u \notin N_\Gamma(w)$ , the degree of  $w$  in  $\Gamma - u$  is  $r$ . Similarly, the degree of  $w'$  in  $\Gamma - u$  is  $s$ . Since  $w, w' \notin N_\Gamma(u)$ , any  $u' \in N_\Gamma(u)$  has degree  $k - 1$  in the graph  $\Gamma - u$ . Finally, any  $u' \notin \{w, w'\} \cup N_\Gamma(u)$  has degree  $k$  in the graph  $\Gamma - u$ . Therefore, the degree sequence of  $\Gamma - u$  is

$$(r, s, \underbrace{k-1, \dots, k-1}_k, \underbrace{k, \dots, k}_{|V\Gamma|-k-3}).$$

By a similar argument, taking into account the fact that  $w, w' \in N_\Gamma(v)$ , the degree sequence of  $\Gamma - v$  is

$$(r-1, s-1, \underbrace{k-1, \dots, k-1}_{k-2}, \underbrace{k, \dots, k}_{|V\Gamma|-k-1}).$$

Since  $\Gamma - u \cong \Gamma - v$ , these degree sequences must be equal. Hence, we must have  $r = s = k$ , so  $\Gamma$  is regular. This contradicts [Lemma 3.1](#). Therefore, it follows that  $v \notin N_\Gamma(w')$ .

Now, by a similar degree argument, we claim that  $r = k$ . By the previous argument, the degree sequence of  $\Gamma - u$  is

$$(r, s, \underbrace{k-1, \dots, k-1}_k, \underbrace{k, \dots, k}_{|V\Gamma|-k-3}).$$

Moreover, taking into account the fact that  $v \in N_\Gamma(w)$  and  $v \notin N_\Gamma(w')$ , it follows that the degree sequence of  $\Gamma - v$  is

$$(r-1, s, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}).$$

Since  $\Gamma - u \cong \Gamma - v$ , these degree sequences must be equal. This is only possible if  $r = k$ , proving our claim.

Now suppose that there exists  $v' \in A$  such that  $v' \notin N_\Gamma(w)$  and  $v' \in N_\Gamma(w')$ . By the symmetric argument,  $s = k$ , which implies that  $\Gamma$  is regular. This contradicts [Lemma 3.1](#). Moreover, since there cannot exist  $v' \in A$  such that  $v' \in N_\Gamma(w)$  and  $v' \in N_\Gamma(w')$ , we must conclude that

$$N_\Gamma(w') \setminus \{w\} = \emptyset.$$

There are now two cases to consider.

- In the case  $N_\Gamma(w') = \emptyset$ , the vertex  $w'$  is an isolated vertex in  $\Gamma$ . It follows that the vertices in  $A$  are mutually pseudo-similar in  $\Gamma$  if and only if they are mutually pseudo-similar in the graph  $\Gamma - w'$ . By [Theorem 3.3](#), the vertices in  $A$  are not mutually pseudo-similar in  $\Gamma - w'$ , therefore they are not mutually pseudo-similar in  $\Gamma$ . This is a contradiction.
- In the case  $N_\Gamma(w') = \{w\}$ , the graph  $\Gamma$  is of the form in [Figure 4](#). There are three sub-cases to be considered, depending on the value of  $k$ .
  1. In the case  $k = 1$ , each  $u \in A$  is adjacent to only  $w$ . Therefore,  $\Gamma$  is a star graph with center  $w$ . In this case, all the vertices in  $A$  are similar, which is a contradiction.
  2. In the case  $k = 2$ , each vertex  $u \in A$  is adjacent to  $w$  and exactly one  $v \in A$ . Since the same is true of the vertex  $v$ , it follows that there is an automorphism of  $\Gamma$  exchanging  $u$  and  $v$ . Therefore,  $u \sim_S v$ , which is a contradiction. This is illustrated in [Figure 5](#).

3. In the case  $k > 2$ , suppose that  $u, v \in A$  and

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. In the graph  $\Gamma - u$ , the vertex  $w'$  has degree 1,  $w$  has degree  $|A| + 1 > 1$ , each  $u' \in N_\Gamma(u)$  has degree  $k - 1 > 1$ , and the remaining vertices have degree  $k > 2$ . A similar observation is true for the graph  $\Gamma - v$ . It follows that  $\varphi$  fixes  $w'$ , and since

$$N_{\Gamma-u}(w') = N_{\Gamma-v}(w') = \{w\},$$

$\varphi$  fixes  $w$ . Consider the image of  $N_\Gamma(u)$  under  $\varphi$ . By a degree argument,  $\varphi$  induces the partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Since  $\varphi$  fixes  $w$ , it then follows that  $\varphi$  induces the partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Therefore,  $\varphi$  can be extended to an automorphism of  $\Gamma$  mapping  $u$  to  $v$ . This is a contradiction.

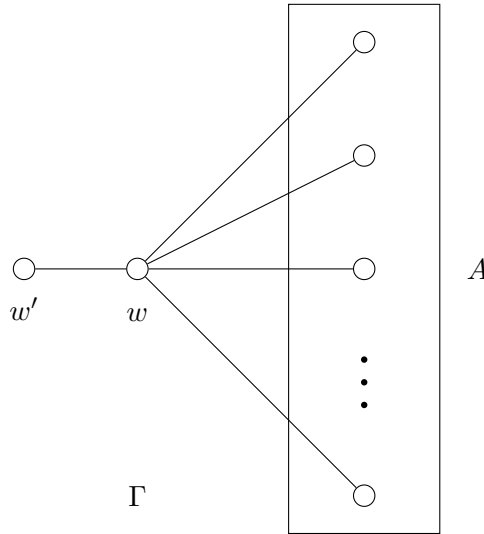


Figure 4: The graph  $\Gamma$  in the case  $N_\Gamma(w') = \{w\}$ .

Therefore, we have demonstrated that, in all cases,  $N_\Gamma(w') \setminus \{w\} = \emptyset$  leads to a contradiction. The only conclusion left is that our initial assumption that there exists  $u \in A$  such that  $u \notin N_\Gamma(w)$  and  $u \notin N_\Gamma(w')$  is false. Then, the fact that there cannot exist  $u \in A$  such that  $u \in N_\Gamma(w)$  and  $u \in N_\Gamma(w')$  can then be deduced from considering the complement  $\tilde{\Gamma}$  of  $\Gamma$ , since  $\Gamma$  and  $\tilde{\Gamma}$  have the same partial automorphism monoid (Corollary 6.5 in [JJSS21]).

Therefore, for every  $u \in A$ , either  $u \in N_\Gamma(w)$  or  $u \in N_\Gamma(w')$ , but not both. This proves Proposition (a).

□

**Proposition (b).**  $r = s = k + 1$ .

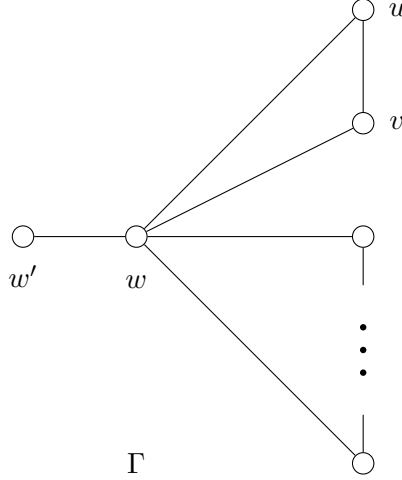


Figure 5: The graph  $\Gamma$  in the case  $N_\Gamma(w') = \{w\}$  and  $k = 2$ .

*Proof.* Let  $u, v \in A$  be such that  $u \in N_\Gamma(w)$  and  $v \in N_\Gamma(w')$ . Since, by Proposition (a),  $u \notin N_\Gamma(w')$  and  $v \notin N_\Gamma(w)$ , it follows that the degree sequence of  $\Gamma - u$  is

$$(r-1, s, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}),$$

and the degree sequence in  $\Gamma - v$  is

$$(r, s-1, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}).$$

Since  $\Gamma - u \cong \Gamma - v$ , it follows that these degree sequences must be equal. Therefore, we must have  $r = s$ .

We now move on to showing that  $r = k + 1$ . The proof is by a degree argument, however the special case where  $r = k - 1$  must be treated separately. Assume, for a contradiction, that  $r \neq k + 1$ .

Let  $u, v \in A$  be such that  $u, v \in N_\Gamma(w)$ . Let

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

be a partial automorphism.

In the graph  $\Gamma - u$ , each  $u' \in N_\Gamma(u) \setminus \{w\}$  has degree  $k - 1$ . Since  $w \in N_\Gamma(u)$ , the vertex  $w$  has degree  $r - 1$  in the graph  $\Gamma - u$ , and since  $w' \notin N_\Gamma(u)$ , the vertex  $w'$  has degree  $r$  in the graph  $\Gamma - u$ . The remaining vertices all have degree  $k$  in  $\Gamma - u$ . A similar observation is true for the graph  $\Gamma - v$ .

Suppose first that  $r \neq k - 1$ . Since  $r \neq k + 1$ , the vertex  $w$  is the unique vertex of degree  $r - 1$  in the graph  $\Gamma - u$  and in the graph  $\Gamma - v$ . It follows that  $w\varphi = w$ . Moreover, since  $r \neq k - 1$ , the vertices in  $N_\Gamma(u) \setminus \{w\}$  are only vertices in the graph  $\Gamma - u$  of degree  $k - 1$ . A similar observation is true of  $N_\Gamma(v) \setminus \{w\}$  in the graph  $\Gamma - v$ . Therefore,  $\varphi$  induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Thus, combining these two facts,  $\varphi$  induces a partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and therefore can be extended to an automorphism of  $\Gamma$  mapping  $u$  to  $v$ . This is a contradiction.

We now consider the special case when  $r = k - 1$ . First, in both  $\Gamma - u$  and  $\Gamma - v$ , the vertex  $w$  is the unique vertex of degree  $k - 2$ , and so  $w\varphi = w$ . It follows that  $\varphi$  maps the set  $N_{\Gamma-u}(w)$  onto the set  $N_{\Gamma-v}(w)$ . That is,  $\varphi$  induces a partial automorphism

$$\theta : N_\Gamma(w) \setminus \{u\} \rightarrow N_\Gamma(w) \setminus \{v\}.$$

Moreover, each  $u' \in N_\Gamma(w') \setminus \{w\}$  has degree  $k$  in the graph  $\Gamma - u$ . Since  $N_\Gamma(w') \setminus \{w\}$  is disjoint to both the sets  $N_\Gamma(w) \setminus \{u\}$  and  $N_\Gamma(w) \setminus \{v\}$ , and moreover since  $w\varphi = w$  and  $\deg_{\Gamma-u}(w') = \deg_{\Gamma-v}(w') = k - 1$ , it follows that  $N_\Gamma(w') \setminus \{w\}$  is fixed setwise by  $\varphi$ .

Now consider the image of  $w'$  under  $\varphi$ . Since  $r = k - 1$ , the set of vertices of degree  $k - 1$  in the graph  $\Gamma - u$  is

$$\{w'\} \cup (N_\Gamma(u) \setminus \{w\}).$$

Similarly, the set of vertices of degree  $k - 1$  in the graph  $\Gamma - v$  is

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w\}).$$

Therefore, by a degree argument,  $\varphi$  induces a partial automorphism

$$\psi : \{w'\} \cup (N_\Gamma(u) \setminus \{w\}) \rightarrow \{w'\} \cup (N_\Gamma(v) \setminus \{w\}).$$

Using the fact that  $w\varphi = w$ , it follows that  $\varphi$  induces a partial automorphism

$$\psi' : \{w'\} \cup N_\Gamma(u) \rightarrow \{w'\} \cup N_\Gamma(v).$$

Let  $u' \in \{w'\} \cup N_\Gamma(u)$  be such that  $u'\varphi = w'$ . If  $u' = w'$  then we are done. Otherwise,  $w'\varphi \in N_\Gamma(v)$ , and so the image of  $N_\Gamma(u)$  under  $\varphi$  is

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w'\varphi\}).$$

Consider the neighbours of  $w'$  in the graph  $\Gamma - u$ .

If  $w$  and  $w'$  are adjacent, then

$$N_{\Gamma-u}(w') = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

It follows, since  $w$  is fixed by  $\varphi$  and  $N_\Gamma(w') \setminus \{w\}$  is fixed setwise by  $\varphi$ , that

$$N_{\Gamma-v}(w'\varphi) = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

Moreover,

$$N_{\Gamma-u}(w') = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

Similarly, in the case that  $w$  and  $w'$  are not adjacent,

$$N_{\Gamma-u}(w') = N_\Gamma(w') = N_\Gamma(w') \setminus \{w\}$$

which is fixed setwise by  $\varphi$ , and so

$$N_{\Gamma-v}(w'\varphi) = N_{\Gamma}(w') \setminus \{w\} = N_{\Gamma-v}(w').$$

Therefore,  $\Gamma - v$  has an automorphism which exchanges  $w'$  and  $w'\varphi$ , and fixes every other vertex pointwise. This is an automorphism of  $\Gamma$  mapping

$$\{w'\} \cup (N_{\Gamma}(v) \setminus \{w'\varphi\}) = (N_{\Gamma}(u))\varphi$$

to  $N_{\Gamma}(v)$ , and hence, by [Theorem 2.8](#),  $u$  and  $v$  are similar in  $\Gamma$ . This is a contradiction.

Therefore, we must conclude that  $r = s = k + 1$ . This proves Proposition (b). □

**Proposition (c).** Each  $u \in N_{\Gamma}(w) \setminus \{w'\}$  has at least one neighbour in  $N_{\Gamma}(w') \setminus \{w\}$ .

*Proof.* Let  $u \in A$ . Since  $N_{\Gamma}(w) \cap N_{\Gamma}(w') = \emptyset$  and  $(N_{\Gamma}(w) \cup N_{\Gamma}(w')) \setminus \{w, w'\} = A$ , we can partition the set  $N_{\Gamma}(u) \setminus \{w\}$  as follows

$$N_{\Gamma}(u) \setminus \{w\} = N_{\Gamma}^w(u) \cup N_{\Gamma}^{w'}(u),$$

where  $N_{\Gamma}^w(u)$  is the set of neighbours of  $u$  in  $A$  which are adjacent to  $w$ , and  $N_{\Gamma}^{w'}(u)$  is the set of neighbours of  $u$  in  $A$  which are adjacent to  $w'$ .

Let  $u, v \in N_{\Gamma}(w) \setminus \{w'\}$ , and suppose that

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. In the graph  $\Gamma - u$ , each  $u' \in N_{\Gamma}(u) \setminus \{w\}$  has degree  $k - 1$ , whereas every vertex not in this set has degree at least  $k$ . A similar observation is true in the graph  $\Gamma - v$ . Therefore, by a degree argument,  $\varphi$  induces a partial automorphism

$$\psi : N_{\Gamma}(u) \setminus \{w\} \rightarrow N_{\Gamma}(v) \setminus \{w\}.$$

Consider  $w'\varphi$ . Since  $w' \notin N_{\Gamma}(u)$ , the vertex  $w'$  has degree  $k + 1$  in the graph  $\Gamma - u$ . Every other vertex in  $\Gamma - u$  has degree  $k$  or  $k - 1$ . A similar observation is true of  $w'$  in the graph  $\Gamma - v$ . Therefore, by a degree argument,  $w'\varphi = w'$ .

Since  $\varphi$  fixes  $w'$ , it follows that  $\varphi$  maps the set  $N_{\Gamma-u}(w')$  onto the set  $N_{\Gamma-v}(w')$ , and hence the set

$$N_{\Gamma}(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$

is fixed setwise. Therefore,  $\varphi$  must induce partial automorphisms

$$\begin{aligned} \theta : N_{\Gamma}^w(u) &\rightarrow N_{\Gamma}^w(v) \\ \theta' : N_{\Gamma}^{w'}(u) &\rightarrow N_{\Gamma}^{w'}(v). \end{aligned}$$

Hence,  $N_{\Gamma}^w(u) \cong N_{\Gamma}^w(v)$  and  $N_{\Gamma}^{w'}(u) \cong N_{\Gamma}^{w'}(v)$ , and so

$$\begin{aligned} |N_{\Gamma}^w(u)| &= |N_{\Gamma}^w(v)| \\ |N_{\Gamma}^{w'}(u)| &= |N_{\Gamma}^{w'}(v)|. \end{aligned}$$

Since  $u$  and  $v$  were arbitrary vertices in  $N_\Gamma(w) \setminus \{w'\}$ , it follows that there exist integers  $p$  and  $q$  such that, for all  $u \in N_\Gamma(w) \setminus \{w'\}$ ,

$$|N_\Gamma^w(u)| = p \text{ and } |N_\Gamma^{w'}(u)| = q.$$

By the symmetric argument, with  $u, v \in N_\Gamma(w') \setminus \{w\}$ , it also follows that, for all  $u \in N_\Gamma(w') \setminus \{w\}$ ,

$$|N_\Gamma^{w'}(u)| = p \text{ and } |N_\Gamma^w(u)| = q.$$

Now suppose that  $q = 0$ . We distinguish between the case when  $w, w'$  are adjacent and when  $w, w'$  are not adjacent.

- In the case where  $w, w'$  are not adjacent, then  $\Gamma$  is not connected. The two connected components of  $\Gamma$  are  $N_\Gamma(w) \cup \{w\}$  and  $N_\Gamma(w') \cup \{w'\}$ . This is illustrated in [Figure 6](#). Consider, in particular, the component  $N_\Gamma(w) \cup \{w\}$ . The vertices in  $N_\Gamma(w)$  are mutually pseudo-similar in  $\Gamma$  if and only if they are mutually pseudo-similar in the graph

$$\Gamma[N_\Gamma(w) \cup \{w\}].$$

By [Theorem 3.3](#), they cannot be mutually pseudo-similar in the graph  $\Gamma[N_\Gamma(w) \cup \{w\}]$ , hence they are not mutually pseudo-similar in  $\Gamma$ . This is a contradiction.

- In the case where  $w, w'$  are adjacent, first notice that we must have  $|A| \geq 4$ , so that  $|N_\Gamma(w) \setminus \{w'\}| = |N_\Gamma(w') \setminus \{w\}| \geq 2$ . This is illustrated in [Figure 7](#). Otherwise,  $\Gamma$  is a path graph with an automorphism exchanging  $N_\Gamma(w) \setminus \{w'\}$  and  $N_\Gamma(w') \setminus \{w\}$ , making the vertices in  $A$  similar, which is a contradiction.

Let  $u, v \in N_\Gamma(w) \setminus \{w'\}$ , and let

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

be a partial automorphism. In the graph  $\Gamma - u$ , the vertex  $w'$  has degree  $k+1$ ; each  $u' \in N_\Gamma(w')$  has degree  $k$ ; the vertex  $w$  has degree  $k$ ; each  $u' \in N_\Gamma(w) \setminus (N_\Gamma(u) \cup \{w'\})$  has degree  $k$ ; and each  $u' \in N_\Gamma(u) \setminus \{w\}$  has degree  $k-1$ . A similar observation is true for the graph  $\Gamma - v$ . It follows that  $\varphi$  induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Moreover, since  $w'$  is the only vertex of degree  $k+1$  in both  $\Gamma - u$  and  $\Gamma - v$ , it follows that  $w'\varphi = w'$ . Hence,  $\varphi$  fixes

$$N_\Gamma(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$

setwise. Consider  $w\varphi$ . Since  $w \in N_\Gamma(w')$ , it follows that

$$w\varphi \in N_\Gamma(w').$$

Since  $N_\Gamma(w) \geq 2$ , the vertex  $w$  has a neighbour  $u' \in N_\Gamma(w) \setminus \{w'\}$  which has degree  $k-1$  in the graph  $\Gamma - u$ . A similar observation is true for the graph  $\Gamma - v$ . In contrast, since  $q = 0$ , each  $u' \in N_\Gamma(w') \setminus \{w\}$  has no neighbour in the set  $N_\Gamma(w) \setminus \{w'\}$ , therefore each neighbour of  $u'$  has degree  $k$  or  $k+1$  in the graph  $\Gamma - u$ . Similarly in the graph  $\Gamma - v$ . It follows that, if  $\varphi$  is a partial automorphism,  $w\varphi = w$ , and therefore  $\varphi$  induces the partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Hence,  $\varphi$  can be extended to an automorphism of  $\Gamma$  mapping  $u$  to  $v$ , making  $u$  and  $v$  similar. This is a contradiction.

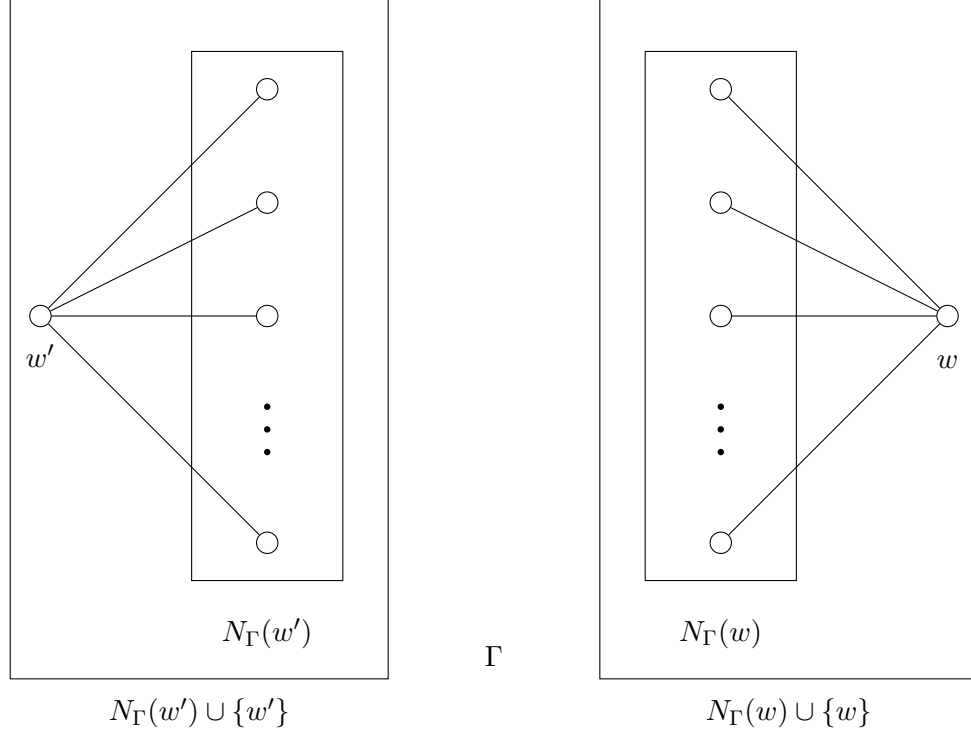


Figure 6: The graph  $\Gamma$  in the case that  $w, w'$  are not adjacent.

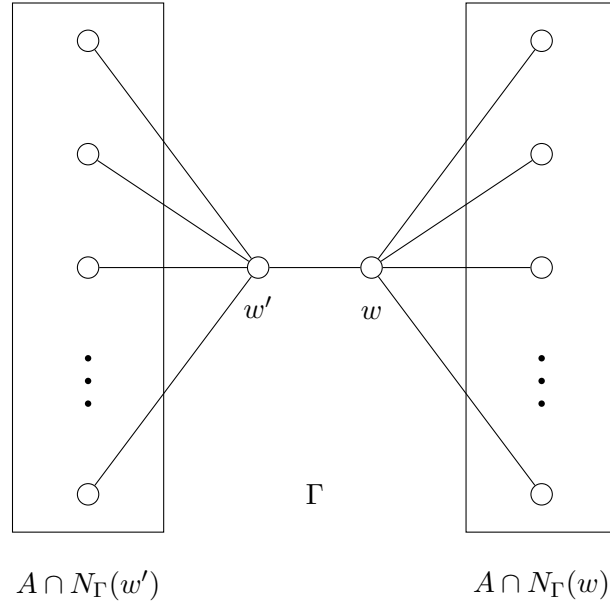


Figure 7: The graph  $\Gamma$  in the case that  $w, w'$  are adjacent.

Therefore,  $q \geq 1$ , so every  $u \in N_{\Gamma}(w) \setminus \{w'\}$  has at least one neighbour in  $N_{\Gamma}(w') \setminus \{w\}$ , as desired. This proves Proposition (c).

**Proposition (d).** Each  $u \in N_{\Gamma}(w') \setminus \{w\}$  has at least one neighbour in  $N_{\Gamma}(w) \setminus \{w'\}$ .

Continuing from the proof of Proposition (c), suppose instead that  $p = 0$ . Let  $u, v \in N_\Gamma(w) \setminus \{w'\}$ , and suppose that

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. Once again, by a degree argument  $\varphi$  fixes  $w'$ , and hence fixes  $N_\Gamma(w')$  setwise. Moreover, by a degree argument,  $\varphi$  induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

- In the case  $w, w'$  are not adjacent, since  $N_\Gamma(w')$  is fixed setwise, it follows that  $N_\Gamma(w) \setminus \{u\}$  is mapped onto  $N_\Gamma(w) \setminus \{v\}$ , and hence  $w$  is fixed. It follows that  $\varphi$  induces a partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and so  $\varphi$  can be extended to an automorphism of  $\Gamma$  mapping  $u$  to  $v$ . This implies  $u \sim_S v$  in  $\Gamma$ , which is a contradiction.

- In the case  $w, w'$  are adjacent, since  $N_\Gamma(w')$  is fixed setwise, it follows that  $w\varphi \in N_\Gamma(w')$ . Moreover, by a degree argument,  $w\varphi \in \{w\} \cup (N_\Gamma(w') \setminus N_\Gamma(v))$  in the graph  $\Gamma - v$ . In the graph  $\Gamma - v$ , the vertex  $w$  is adjacent to  $w'$  and all of the  $k - 1$  vertices in the set  $N_\Gamma(w) \setminus \{v\}$ . Moreover, in the graph  $\Gamma - v$  every  $v' \in N_\Gamma(w') \setminus N_\Gamma(v)$  is adjacent to  $w'$  and all of the  $k - 1$  vertices in the set  $N_\Gamma(w) \setminus \{v, w'\}$ . Hence,

$$N_{\Gamma-v}(w) = N_{\Gamma-v}(w\varphi),$$

and therefore  $\Gamma - v$  has an automorphism which exchanges  $\varphi$  and  $w\varphi$ , and fixes all other vertices pointwise. This is an automorphism of  $\Gamma - v$  mapping the set  $(N_\Gamma(u))\varphi$  onto  $N_\Gamma(v)$ , and so by [Theorem 2.8](#),  $u \sim_S v$  in  $\Gamma$ . This is a contradiction.

It follows that  $p \geq 1$ , and so each  $u \in N_\Gamma(w') \setminus \{w\}$  has at least one neighbour in the set  $N_\Gamma(w') \setminus \{w\}$ . This proves Proposition (d).  $\square$

*Proof of Theorem 2.9.* For the remainder of the argument, let  $u, v \in A$  be such that  $u, v \in N_\Gamma(w)$ . Suppose that

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. By Claim 2, the vertex  $w'$  has degree  $k + 1$  in the graph  $\Gamma - u$ , and also has degree  $k + 1$  in the graph  $\Gamma - v$ . Every vertex not equal to  $w'$  has degree at most  $k$  in  $\Gamma - u$ , and similarly in  $\Gamma - v$ . Thus,  $w'\varphi = w'$ . It follows that  $\varphi$  maps the set  $N_{\Gamma-u}(w')$  onto the set  $N_{\Gamma-v}(w')$ , and therefore

$$N_\Gamma(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$

is fixed setwise.

Moreover, every  $u' \in N_\Gamma(u) \setminus \{w\}$  has degree  $k - 1$  in  $\Gamma - u$ . Similarly, every  $v' \in N_\Gamma(v)$  has degree  $k - 1$  in  $\Gamma - v$ . The remaining vertices all have degree at least  $k$ . It follows, by a degree argument, that  $\varphi$  induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Now consider  $w\varphi$ . By Proposition (b),  $w$  has degree  $k + 1$  in  $\Gamma$ , and so has degree  $k$  in  $\Gamma - u$ . Similarly,  $w$  has degree  $k$  in  $\Gamma - v$ . Therefore, by a degree argument, either

$$w\varphi \in N_\Gamma(w'),$$

or

$$w\varphi \in \{w\} \cup N_\Gamma(w) \setminus (\{w'\} \cup N_\Gamma(v)).$$

We again distinguish between the cases when  $w, w'$  are adjacent and  $w, w'$  are not adjacent.

- In the case  $w, w'$  are not adjacent, then  $w\varphi \in \{w\} \cup N_\Gamma(w) \setminus N_\Gamma(v)$ . Suppose that  $w\varphi \neq w$ . By Claim 3, every  $u' \in N_\Gamma(w) \setminus \{w'\}$  has at least one neighbour in the set  $N_\Gamma(w') \setminus \{w\}$ . It follows that  $w\varphi$  has at least one neighbour in  $N_\Gamma(w') \setminus \{w\}$ . However,  $w$  has no neighbour in  $N_\Gamma(w') \setminus \{w\}$ , and since  $N_\Gamma(w')$  is fixed setwise by  $\varphi$ , this is a contradiction.
- In the case  $w, w'$  are adjacent, then  $w\varphi \in N_\Gamma(w)$ . Suppose for a contradiction that  $w\varphi \neq w$ . In  $\Gamma - u$ , the vertex  $w$  has no neighbours in the set  $N_\Gamma(w') \setminus \{w\}$ . However, by Claim 3, if  $w\varphi \in N_\Gamma(w') \setminus \{w\}$ , then  $w\varphi$  has at least one neighbour in the set  $N_\Gamma(w') \setminus \{w\}$ . Denote this neighbour  $u'$ . Since  $\varphi$  fixes  $N_\Gamma(w')$  setwise, it follows that  $u'\varphi^{-1} \in N_\Gamma(w')$ . Moreover, since  $w\varphi$  and  $u'$  are adjacent, it follows that  $u' \neq w\varphi$ , and hence  $u'\varphi^{-1} \neq w$ . Therefore,  $u'\varphi^{-1} \in N_\Gamma(w') \setminus \{w\}$ , so  $w$  has a neighbour in the set  $N_\Gamma(w') \setminus \{w\}$ . This contradicts Proposition (a).

Therefore,  $w\varphi = w$ . It follows that  $\varphi$  induces the partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and hence by Lemma 2.7,  $u \sim_S v$  in  $\Gamma$ . This contradicts their pseudo-similarity. It follows that there cannot exist a graph  $\Gamma$  containing a set of mutually pseudo-similar vertices of size  $|V\Gamma| - 2$ . This proves Theorem 3.4. □

The method of reduction used to prove Theorem 3.4 is clearly limited. As we decrease  $|A|$  relative to  $|V\Gamma|$ , the amount of case analysis required rapidly becomes untenable. In fact, the method will eventually fail entirely: there exists a graph  $\Gamma$  with a set of mutually pseudo-similar vertices of size  $|V\Gamma| - 6$ . For an example, see Figure 9.

One possible route for further progress may be to consider generalisations of Proposition (a). If such a generalisation could be used to show that, for each  $u \in A$ , a sufficiently large fraction of the vertices in  $N_\Gamma(u)$  are not pseudo-similar to  $u$  in  $\Gamma$ , then this could be used to prove a new lower bound for Question 1.3.

### 3.2 Digraphs, and constructing graphs with arbitrarily large sets of mutually pseudo-similar vertices

The case when  $\Gamma$  is digraph contrasts sharply with the case when  $\Gamma$  is a graph. For example, Theorem 3.2 is false for digraphs: there exist digraphs with every vertex mutually pseudo-similar to every other vertex. An important example is the *transitive tournament*,  $T_k$ , defined on the vertex set

$$VT_k = \{1, 2, \dots, k\}$$

with  $u \rightarrow v$  if and only if  $u < v$ . For a picture, see Figure 8.

**Proposition 3.5.** *In  $T_k$ , every vertex is mutually pseudo-similar to every other vertex.*

*Proof.* First, note that  $T_k$  has no automorphism other than the identity, so no pair of distinct vertices  $u, v \in VT_k$  is similar. By considering  $T_k$  as a linear order on the vertices  $\{1, 2, \dots, k\}$ , we can see that  $T_k - u \cong T_{k-1}$  for every  $u \in VT_k$ . Therefore, for every pair  $u, v \in T_k$ , we have  $T_k - u \cong T_k - v$ . Thus, any pair  $u, v \in T_k$  is pseudo-similar.  $\square$

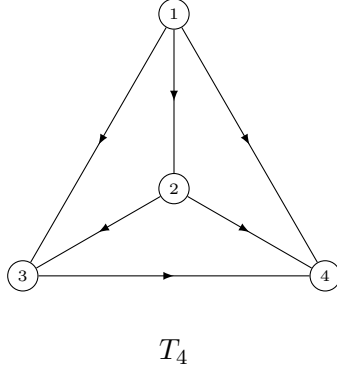


Figure 8: The transitive tournament  $T_4$ .

Given a transitive tournament  $T_k$ , it is possible to replace the directed edges with undirected ‘gadgets’ while maintaining the mutual pseudo-similarity of the vertices. We describe on such construction, due to [KJSS81]. Consider the undirected gadget  $\Gamma$  in Figure 9. The vertices  $u$  and  $v$  are pseudo-similar in  $\Gamma$ , and, moreover, are cutvertices of  $\Gamma$ , that is, the graphs  $\Gamma - u$  and  $\Gamma - v$  are not connected. For each pair  $u, v \in VT_k$ , replace the directed edge  $u \rightarrow v$  with the gadget  $\Gamma$ , including the extra vertices on each side. See Figure 10 for an example. This produces a sequence of graphs,  $\Gamma_k$ , containing a set of  $k$  mutually pseudo-similar vertices and

$$|VT_k| = k + (|VT| - 2) \binom{k}{2} = k + 6 \binom{k}{2} = \mathcal{O}(k^2).$$

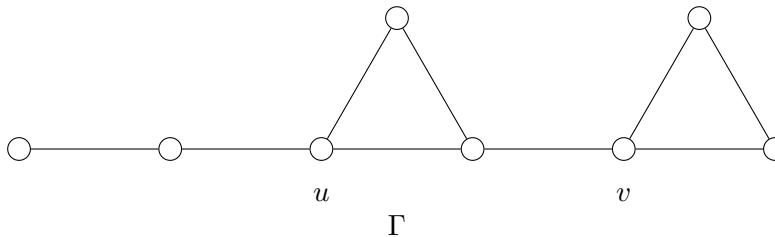


Figure 9: The gadget  $\Gamma$ , with pseudo-similar cutvertices  $u$  and  $v$ .

The problem with transforming  $T_k$  into an undirected graph in this manner is that, while every vertex in  $T_k$  is mutually pseudo-similar to every other vertex, and hence contains a set of  $k$  mutually pseudo-similar vertices,  $T_k$  contains  $\binom{k}{2} = \mathcal{O}(k^2)$  edges. When replacing a directed edge with a gadget, at least one more vertex is added, so the resulting undirected graph has  $\mathcal{O}(k^2)$  vertices.

A natural question is therefore to ask, does there exist a sequence of digraphs with every vertex mutually pseudo-similar to every other vertex and with fewer than  $\mathcal{O}(k^2)$  directed edges?

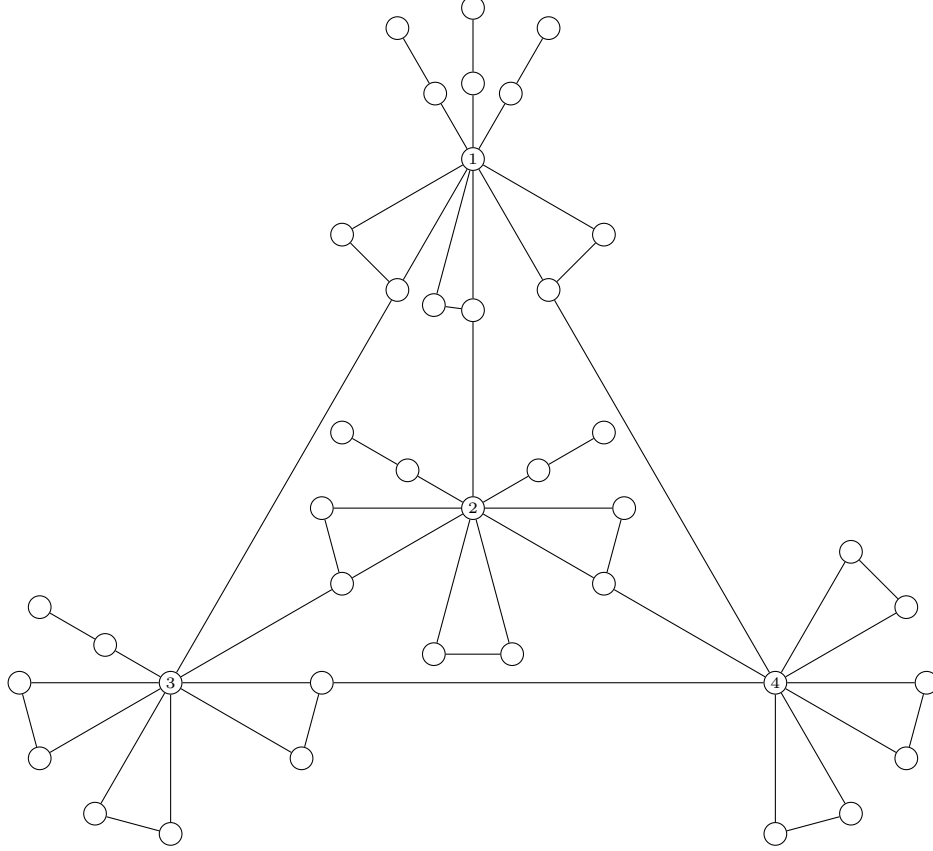


Figure 10: The transitive tournament  $T_4$ , with directed edges replaced with appropriate undirected gadgets.

**Problem 3.6.** Let  $\Gamma_k$  be a digraph with  $|V\Gamma_k| = k$  such that every vertex is mutually pseudo-similar to every other vertex. Does this then imply  $\Gamma_k \cong T_k$ ?

This problem is deceptively difficult. One subtlety occurs when considering extensions of partial automorphisms of the form

$$\varphi : \Gamma_k - u \rightarrow \Gamma_k - v$$

to automorphisms of  $\Gamma_k$ . Unlike in graphs, it is not sufficient for  $\varphi$  to be able to be extended to an automorphism of  $\Gamma_k$  that  $\varphi$  maps the (undirected) neighbours of  $u$  to the (undirected) neighbours of  $v$ . It also required that  $\varphi$  respects the partition of  $N_\Gamma(u)$  and  $N_\Gamma(v)$  into the in-neighbours and out-neighbours. This subtlety causes degree arguments of the kind used in [Lemma 3.1](#) to lose their power.

Consider, for instance, the following proposition about extending partial automorphisms of digraphs, which may be viewed as an attempt at an analogue of [Lemma 2.7](#) for digraphs.

**Proposition 3.7.** Let  $\Gamma$  be a digraph, and  $u, v \in V\Gamma$  be removal similar. Suppose that  $\Gamma - u - v$  has no automorphism mapping an in-neighbour of  $u$  to an out-neighbour of  $u$ , and vice-versa. Let  $\varphi : \Gamma - u \rightarrow \Gamma - v$  be a partial automorphism, and suppose that restricting  $\varphi$  induces two partial

automorphisms

$$\begin{aligned}\psi &: N_\Gamma(u) \rightarrow N_\Gamma(v) \\ \theta &: N'_\Gamma(v) \rightarrow N'_\Gamma(u)\end{aligned}$$

such that  $v\theta = u$ . Then  $u$  and  $v$  are similar in  $\Gamma$ .

*Proof.* Let  $\Gamma$ ,  $u$  and  $v$  be as given in the statement of the lemma, and let  $\varphi : \Gamma - u \rightarrow \Gamma - v$  be a partial automorphism which induces two partial automorphisms

$$\begin{aligned}\psi &: N_\Gamma(u) \rightarrow N_\Gamma(v) \\ \theta &: N'_\Gamma(v) \rightarrow N'_\Gamma(u)\end{aligned}$$

Notice that, since  $\varphi : \Gamma - u \rightarrow \Gamma - v$ , and  $v\varphi = u$ , it follows that  $\varphi^2$  is an automorphism of  $\Gamma - u - v$ . Suppose, for a contradiction, that there is a  $w \rightarrow u$  such that  $v \rightarrow w\varphi$ . Then,  $w\varphi^2 = (w\varphi)\theta$ , so if  $v \rightarrow w\varphi$  then  $v\theta \rightarrow (w\varphi)\theta$  and so  $u \rightarrow w\varphi^2$ . Therefore,  $\varphi^2$  is an automorphism of  $\Gamma - u - v$  which maps an in-neighbour of  $u$  to an out-neighbour of  $u$ , which is a contradiction. Since, in addition to this,  $\varphi$  induces the map

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v)$$

it follows that  $\varphi$  must map in-neighbours of  $u$  to in-neighbours of  $v$ , and by the symmetric argument, map out-neighbours of  $u$  onto out-neighbours of  $v$ . It follows that  $\varphi$  can be extended to an automorphism of  $\Gamma$  mapping  $u$  to  $v$ , so  $u \sim_S v$  in  $\Gamma$ .  $\square$

**Proposition 3.2** is weak in the sense that the requirements placed on  $\Gamma$  and  $\varphi$  are very strong. If **Question 3.6** is to be approached, then these conditions must be weakened significantly.

Another approach is to weaken the condition that  $|V\Gamma_k| = k$ . If instead, we only require that  $\Gamma_k$  contains a set of  $k$  mutually pseudo-similar vertices with  $|V\Gamma_k| = \mathcal{O}(k)$ , then a wider class of digraphs may be considered.

**Problem 3.8.** *Construct a sequence of digraphs  $\Gamma_k$  such that*

1.  $\Gamma_k$  contains a set of  $k$  mutually pseudo-similar vertices;
2.  $|V\Gamma_k| = \mathcal{O}(k)$ ; and
3.  $|E\Gamma_k| = \mathcal{O}(k)$ .

If such a sequence  $\Gamma_k$  were found, then it could possibly be transformed using undirected gadgets to produce a sequence of graphs having a set of  $k$  mutually pseudo-similar vertices and  $\mathcal{O}(k)$  vertices.

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